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BOOK REVIEWS

Applied Partial Differential Equations. By J. Ockendon, S. Howison, A. Lacey & A. Movchan. Oxford University Press, 1999. 427 pp. ISBN 0 19 853243 1 (paperback). £25.

Any fluid dynamicist will, of course, be aware that partial differential equations are the natural language for mathematical analyses of a vast range of physical phenomena. This book is thus greatly to be welcomed, as it sets out theory and methods for partial differential equations from the perspective of applied mathematicians who are concerned with the actual application of mathematics to model physics. In the spectrum that runs from rigorous 'ϵ–δ' analysis to phenomenological physics, this book can perhaps best be compared in flavour with various texts on 'mathematical methods for physicists': the motivation and illustrations are drawn from the authors' extensive experience of modelling physical and industrial problems, proofs are generally eschewed, and the focus is on how to obtain analytic results and usable formulae.

The material covered is, however, at a substantially more advanced level than that in typical courses on mathematical methods for physicists. The preface describes the book as suitable for a first-year graduate course, but I doubt that many starting graduates have the breadth of background to appreciate both the detailed sophistication of the analysis and the wide variety of physics underlying the modelling. Moreover, the condensed and interwoven material does not seem to isolate and expose the main ideas with quite the clarity a learner might hope for. I suspect its main use is thus more likely to be as a reference for more advanced students and researchers and as valuable source material for an instructor putting a course together. In this context, it is worth noting that each chapter closes with a wealth of non-trivial exercises (without solutions).

The book opens with two chapters on general first-order quasi-linear differential systems, which introduce basic concepts such as Cauchy data and characteristics, domain of definition, well-posedness, weak solutions and shocks. The third chapter, largely on semilinear second-order scalar equations, provides a link to the following three chapters on hyperbolic, elliptic and parabolic equations. While this might appear a conventional division, the discussion moves well beyond the usual presentation of the wave, Laplace and heat equations to include topics such as hodograph techniques, similarity groups for nonlinear systems, and Riemann–Hilbert problems. A welcome chapter on the important topic of free-boundary problems such as solidification, Hele-Shaw flow and vortex-sheet dynamics focuses on issues of interfacial stability and well-posedness that arise from the boundary motion. Finally, a chapter on non-quasi-linear equations, such as the eikonal equation of geometric optics and the equations for the differential geometry of deformed surfaces, together with a 'rag-bag' chapter on miscellaneous topics, concludes the book. A conscious decision by the authors to omit all discussion of numerical or asymptotic methods might be regretted on grounds of balance, but the information content of the book still already exceeds most texts.

The narrative style is chatty and informal with a strong personal stamp. A liberal lacing of emotive expressions encourages the discerning reader to empathise with the authors' instincts by reading danger signals, recognising red-letter days and seeing analyses as trivial, tedious, delicate or hair-raising. 'Feel' and experience are invaluable tools when attacking new problems, and I applaud the authors' attempts to impart some of theirs along with the technical content. Overall, a very worthwhile read, and I'll be glad to have a copy on my shelf.

J. R. Lister

The Least-Squares Finite Element Method – Theory and Applications in Computational Fluid Dynamics and Electromagnetics. By B. Jiang. Springer, 1998. 418 pp. ISBN 3 540 63934 9. DM 148.

The Finite Element Method (FEM) is a well-established approach to the numerical solution of elliptic boundary value problems. In the case of self-adjoint differential operators, arising for example in linear elasticity and heat diffusion, the FEM is essentially a Rayleigh–Ritz technique that minimizes a functional of the unknown solution in a suitable finite-dimensional approximation subspace. For such problems, the FEM enjoys a best-approximation property, and automatically leads to algebraic problems that are symmetric and positive definite.

Important elliptic problems involving partial differential equations in several variables, such as the velocity–pressure formulation of Stokes flow, are more delicate to solve by means of the FEM. Indeed, the corresponding variational principles typically yield saddle-point optimization problems. As a result, the approximation subspaces of the different unknowns cannot be selected independently, and must satisfy a stability condition known as the inf-sup or Ladyzhenskaya–Babuska–Brezzi (LBB) condition. The LBB condition rules out the use of equal-order finite element interpolation for velocity and pressure in the solution of Stokes flow by means of a mixed FEM. Another consequence of the saddle-point nature of the associated variational principle is that the discrete algebraic problem is indefinite, and thus more difficult to solve.

The classical Rayleigh–Ritz FEM is of course limited to self-adjoint differential operators. Boundary value problems involving non-self-adjoint operators (e.g. convective transport and Navier–Stokes flow) can be solved by means of a Galerkin FEM (GFEM), which is a particular method of weighted residuals. For example, a complete treatment of the GFEM solution of incompressible Newtonian flow is offered in the book by Gresho & Sani (also reviewed in this issue of *JFM*). It is well known that the GFEM is numerically unstable when applied to convection-dominated problems, yielding spurious oscillations in the numerical results that sometimes only disappear at the expense of inordinate mesh refinement.

The above difficulties with standard finite element techniques have motivated the development of alternative approaches. One such approach is the Least-Squares FEM (LSFEM) covered in the present book by Bo-nan Jiang. Basically, the LSFEM amounts to minimizing the L2-norm of the residuals of the equations. This approach offers numerous theoretical and computational advantages. Most notably, the LSFEM always leads to symmetric and positive definite algebraic problems, and is not subject to the LBB stability condition. This is important in practice. Indeed, equal-order interpolation is allowed for all unknowns, and a standard iterative scheme can be used to solve the discretized equations. Furthermore, the LSFEM can be implemented at the element level, without any matrix assembly. Crucial to the practical usefulness of the LSFEM is the transformation of the governing partial differential equations into a first-order system. This is accomplished by introducing additional physically meaningful variables, such as fluxes, vorticity or stresses. This procedure obviously increases the number of unknowns, and requires a proper treatment of boundary conditions. It should also be noted that the transformation of a given problem (e.g. Stokes flow) into a first-order system is not unique, thus yielding different LSFEMs with potentially different numerical properties.

In his book, Bo-nan Jiang offers a unified treatment of the LSFEM. The author covers both mathematical and computational aspects of the method, and illustrates its properties by providing simulation results for a variety of physical problems in fluid mechanics and electromagnetism. The intended readership includes engineers, physicists and researchers having a basic knowledge of the standard FEM for second-order elliptic problems. No special mathematical expertise beyond calculus and elementary differential equations is required, but prior knowledge of fluid mechanics and electromagnetism is implicitly assumed.

This 418-page book is organized in five parts. Part 1 (Chapters 1–3) covers the basics of the LSFEM. The author explains for simple one-dimensional convection problems why the Galerkin method fails while the LSFEM is quite suitable and does not need any stabilizing trick like upwinding. He also compares the LSFEM with the mixed Galerkin method for first-order elliptic systems, and shows why the LSFEM can accommodate equal-order elements. It is Jiang's opinion that the theoretical basis and analysis of the LSFEM is the bounded inverse theorem of linear operators, which would explain why the LSFEM can provide solutions for all types of partial differential equations (not only elliptic) within a single mathematical and computational framework, without any special treatment. Part 2 (Chapters 4–6) deals with the theoretical aspects of the LSFEM applied to linear, first-order systems of partial differential equations. The so-called div-curl, grad-div, and div-curl-grad formulations are discussed in detail. Applications of the LSFEM to fluid mechanics are described in Part 3 (Chapters 7–13). A wide variety of topics is covered, namely inviscid irrotational flows (incompressible and subsonic compressible), steady and transient incompressible viscous flows, convective transport, rotational inviscid flows governed by the incompressible Euler equations, low-speed compressible non-isothermal viscous flows, simulation of two-fluid flows, and high-speed compressible gas flows governed by the Euler equations. Part 4 (Chapter 14) applies the LSFEM to the Maxwell equations of electromagnetism. Finally, Part 5 (Chapter 15) deals with the solution of the discrete equations by means of the element-by-element conjugate gradient iterative method.

I found the book informative, well written, overall decently produced, and certainly useful to readers of *JFM* involved in Computational Fluid Dynamics who wish to apply the LSFEM in their own studies. It should be noted that the book's subject matter includes many recent research results obtained by the author's group, some of which are unpublished elsewhere. The message conveyed in the text thus often reflects the strong author's personal opinions as to the LSFEM and its (more or less?) indisputable advantages over other techniques. At times, the book very much resembles commercial material wherein the LSFEM is presented as the overall best buy available today in the marketplace, which can solve any problem in a robust way and without any special, problem-dependent treatment. My personal opinion is that such a super-method simply does not exist, but this is of course a matter of debate.

Roland Keunings

Incompressible Flow and the Finite Element Method, Vol. 1: Advection-Diffusion and Isothermal Laminar Flow. By P. M. GRESHO & R. L. SANI. John Wiley and Sons, 1998. 1044 pp. ISBN 0 471 96789 0. $320.

While the book by Bo-nan Jiang (see the above review) dismisses in a few paragraphs the classical Galerkin Finite Element Method (GFEM) for solving non-self-adjoint problems such as advection-diffusion and Navier–Stokes flow, the present book by P. M. Gresho & R. L. Sani devotes more than 1000 pages to that very subject!

Indeed, the two books vastly differ in style, content, and opinions. The following quote from Gresho & Sani's preface clearly defines their philosophy:

> There are many ways to 'do' CFD (computational fluid dynamics) today, and there will undoubtedly be more rather than fewer 'tomorrow'. [...W]e shall simply state our own opinion up-front (and 'opinion' it must be, as the jury is still out, and likely to remain so, regarding 'How best to do CFD'): the Galerkin finite element method (GFEM) is one of the good ways to 'do CFD' – especially when flows in or around 'real world' (complex) geometry are of principal interest. Note that 'good' does not necessarily imply easy, or robust. It does, in our view, imply accuracy and generality – and, in some sense, 'honesty'. It is an objective and honest method that tries to remain true to the underlying PDE's (partial differential equations). Hopefully there is still a market for a method that displays these characteristics. There is also a significant market for what we perceive to be less honest methods; namely, those modifying the Galerkin principle in various seemingly ad hoc ways such as 'upwinding' and related stabilizing and artificially dissipative methods. Such methods would be acceptable to us (and, we believe, to many others) if, in addition to the continual demonstration of their more-or-less acknowledged robustness, they would always be used in conjunction with appropriate mesh refinement efforts that would convince both giver and receiver that their final results do represent an accurate approximate solution to the stated problem.

The authors themselves concede that "the scope of this text is both narrow and broad; it is narrow in that it covers only advection-diffusion and isothermal laminar flow, and it is broad because these important 'basic' topics are covered in much details". In fact, a second volume is planned, that would cover coupled transport problems (e.g. buoyancy-driven flows), stability, continuation, and bifurcation analyses, free and moving boundary problems, simulation of turbulent flows, and solution methods for linear and nonlinear algebraic equations.

The book is organized into four chapters and three appendices. A myriad of topics is discussed, whose coverage we can obviously not review in detail in a few paragraphs. Chapter 1 (pages 1–21) introduces the subject matter, mainly by pointing to classical books on incompressible Newtonian flow, to the FEM itself, and its application to fluid mechanics. Although very informative (I liked in particular the many relevant quotes from the cited books), this chapter does not serve as a formal introduction to these topics. Prior knowledge is clearly required. In Chapter 2 (pages 23–356), the authors discuss the application of the GFEM to the linear advection-diffusion equation. Most notably, they display in 'full glory' the semi-discrete differential equations generated by the GFEM for several one- and two-dimensional elements, and offer a very detailed discussion of important topics such as dispersion, phase and group velocity, wiggles, and mesh design. This chapter also describes one of the authors' most important research contributions to this field, namely how to employ local error control to vary the integrator step size. Many numerical examples are also provided. Chapter 3 (pages 357–845) covers the GFEM solution of the incompressible Navier–Stokes equations. It is clearly the most important chapter in the book. As for the advection-diffusion equation, the authors offer a very detailed discussion of the continuum mechanical problem (e.g. many possible formulations, boundary and initial conditions), and give a complete exposition of the GFEM equations. In particular, the important and difficult issues of choice of elements, LBB stability, and pressure modes are covered at length. Solution methods for the semi-discrete time-dependent equations are discussed in the framework of differential-algebraic equations, a unique feature of this book. These topics and many others are illustrated with numerical examples, the most comprehensive being the impulsive start-up flow past a circular cylinder. Finally, Chapter 4 (pages 847–872) describes the *a posteriori* computation of derived quantities like vorticity, heat flux, forces, and moments. In the appendices, the authors develop the one- and two-dimensional finite element local matrices (which is useful for code debugging), they further compare finite elements and finite volumes, and discuss the GFEM in the formal context of projection methods.

This is a unique book, both in style and content. Definitely not an introductory text to the subject matter, it contains a wealth of detailed technical information, much of which is simply not available elsewhere. Its numerous digressions, remarks, quotes, speculations, and rules of thumb are both useful and worth the patience of the reader; they also contribute to the gigantic size of the book. The style is informal; some readers (like myself) will love it, others will hate it. I cannot resist the pleasure of quoting the following excerpts, selected among numerous other candidates in the same vein, which reveal much about the authors' style of writing and opinions on difficult issues.

Regarding the choice of elements (Section 3.13.2), for example, the authors write:

> In this section we shall attempt to summarize the state of confusion (a moving target) regarding element choices, focus on those subsets of elements that we advocate (partly, of course, because of our own experience), and still try to present a reasonably balanced presentation. That this is not entirely possible is probably obvious, since there often seems to be a fairly large increase in adrenalin flow whenever the subject of 'element choices' is discussed. Our discussion will probably also create a few new enemies – a plight we could bear if in addition it attracts enough outsiders and newcomers to give finite elements a try – so that, on balance, the FEM might move forward faster. Our general philosophy will be based on the premise that simplicity is still beautiful, and on the fact that the theory is too often silent. …A colleague recently opined that the entire field would still be in the Stone Age if practitioners had waited for the theorists to prove 'consistency, stability, accuracy, convergence, etc'. We are already clearly in violation of (our understanding of) the 'French school' – for example – which usually seems to require some minimum number of proved theorems before any computer programming and subsequent numerical experiments are permitted. But even they manage to 'ignore' the unfortunate fact that no one has yet been able to prove global existence of solutions to the subject of this book – the NS equations. …C'est la vie.

In Section 2.6, the authors offer the following inspired comments on spurious numerical oscillations:

> Wiggles – the Nemesis of CFD. Perhaps no other single difficulty has generated more frustration and caused more effort than the rapid, high-frequency – typically (but definitely not always) node-to-node (or time-step to time-step) oscillations that come out of the computer and pollute the putative 'solution' than that called, most simply, wiggles. Perhaps no other aspect of CFD has so divided the world into two basic camps: those who hate/fear the wiggles so much that they use only methods that never permit their occurrence, and those who, while not exactly embracing them, believe that there is a message in the wiggles and that there is more to good CFD analyses than simply being wiggle-free. […] The price that is often paid by those who a priori suppress wiggles by their choice of a numerical method is simply that they

> are often solving the wrong problem: i.e., the effectively/numerically much-reduced Peclet or Reynolds number leads the analysts to believe that they are really solving some tough problems when, in fact, they are not (a virtual reality of CFD) because they have changed the problem. The wiggly-camp, on the other hand, often has much difficulty with tough problems wherein, in the worst case, they can get no solution at all. This camp, in which we are fairly firmly (but perhaps not permanently) entrenched, believes that 'The wiggles are telling you something' and try to use wiggle signals as a guide to better mesh design (where possible) or, in the worst cases, admit that the stated problem – truth be told – is just too difficult (for the current generation of computers).[...] While not as bad as turbulence, or pornography, or even art, each of which you may not be able to define but believe that you recognize it when you see it, there really is no pure and simple always-applicable definition of wiggles except perhaps this (which may require some knowledge of physics): wiggles are non-physical oscillations.

The authors' belief is reflected in their new acronym 'GFEMIA: Galerkin Finite Element Method Intelligently Applied', which "requires, besides a lower bound on the analyst's IQ, not much more than common sense – and, of course, an appreciation for some of the subtleties of both fluid mechanics and the numerical methods used to describe it". They also attribute to J. Ferziger the following deep statement: "The greatest disaster one can encounter in computation is not instability or lack of convergence, but results that are good enough to be believable but bad enough to cause trouble".

Considering the length of these (very relevant) digressions, I found it amusing that the authors emphasize the velocity–pressure formulation of the Navier–Stokes equations 'partly owing to space limitations' (page 360)!

This is a great book which I am happy to recommend to the readers of *JFM*. Among its many qualities, honesty is one that is prominent. Indeed, Gresho & Sani never hesitate to play the devil's advocates, by displaying both the positive and negative features of their beloved GFEM. I believe this book will contribute much to the healthy education of future CFD enthusiasts!

Roland Keunings